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LETTER TO THE EDITOR

Explicit path integration on homogeneous spaces

Georg Junker

Physikalisches Institut der Universität Würzburg, Am Hubland, 8700 Würzburg, Federal Republic of Germany

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Abstract. The path integral for the free quantum motion on an arbitrary homogeneous space $\mathscr{M}$ is considered. We expand the short-time propagator in unitary irreducible representations of the transformation group on $\mathscr{M}$. The path integral is performed explicitly by using the orthogonality of the representations. The correct normalised wavefunctions are given by associate spherical functions and the energy spectrum is obtained from the time derivative of the Fourier coefficients of the expansion.

The path integral for the quantum propagator on symmetric spaces has attracted much attention in recent years. The quantum dynamics on the manifold of simple compact and non-compact Lie groups has been discussed extensively [1, 2]. Other problems of interest are related to the quantum propagation on spaces with constant positive [2-4] and negative [4, 5] curvature. For example, quantum chaos is studied for the free particle moving on a manifold of constant negative curvature [5]. Therefore, it is certainly important to study the dynamics in curved spaces [6]. The quantisation of such systems may be achieved by the method of Feynman [7].

In this letter we present a general prescription for the path integration on a homogeneous space $\mathscr{M}$, which may be viewed as a coset space G/H. Here G is the transformation group acting on $\mathscr{M}$, i.e. $\boldsymbol{x}\to\boldsymbol{x}'=g\boldsymbol{x}$, where g is a group element of the transformation group G. H is the stability group of a fixed point $\boldsymbol{a}$ on $\mathscr{M}$, $h\boldsymbol{a}=\boldsymbol{a}$, $\forall h\in \mathrm{H}$. The path integration is performed explicitly for an arbitrary homogeneous space. We will obtain a general formula for the energy eigenvalues and also the normalised wavefunctions. As an example, we consider the Klein-Gordon propagator in an n-dimensional flat space.

Our starting point is the path integral representation of the transition amplitude for a free quantum system on a homogeneous space $\mathscr{M}$ evolving from initial state $|\boldsymbol{x}_a\rangle$ to the final configuration $|\boldsymbol{x}_b\rangle$ in the time $T=t_b-t_a$. In the usual sliced time basis, $T=N\varepsilon$, this propagator is given by

$$K(\boldsymbol{x}_b,\boldsymbol{x}_a,T)=\lim_{N\to\infty}\int\prod_{j=1}^{N}K(\boldsymbol{x}_j,\boldsymbol{x}_{j-1},\varepsilon)\prod_{j=1}^{N-1}\mathrm{d}\boldsymbol{x}_j. \tag{1}$$

We have adopted the standard notation: $\boldsymbol{x}_j=\boldsymbol{x}(t_j)$, $\boldsymbol{x}_b=\boldsymbol{x}_N$ and $\boldsymbol{x}_a=\boldsymbol{x}_0$. $\mathrm{d}\boldsymbol{x}$ is the invariant measure on $\mathscr{M}$. For compact spaces we have $\int_{\mathscr{M}}\mathrm{d}\boldsymbol{x}=|\mathscr{M}|$, where $|\mathscr{M}|$ is the volume of $\mathscr{M}$. In Feynman's path integral approach the short-time propagator is assumed to be given in the semiclassical form [7]

$$K(\boldsymbol{x}_j,\boldsymbol{x}_{j-1},\varepsilon)=\left(\det\left|\frac{\mathrm{i}}{2\pi\hbar}\frac{\partial^2S_j}{\partial\boldsymbol{x}_{j-1}\partial\boldsymbol{x}_j}\right|\right)^{1/2}\exp\{(\mathrm{i}/\hbar)S_j\}$$

where the classical action S_j along the short-time interval ε is approximated by

$$S_j = \int_{t_{j-1}}^{t_j} L(\dot{x}, x)\, \mathrm{d}t \simeq L(\Delta x_j/\varepsilon, \bar{x}_j)\varepsilon \qquad \Delta x_j = x_j - x_{j-1} \quad \bar{x}_j = \tfrac{1}{2}(x_j + x_{j-1}).$$

In the Hamiltonian path integral formulation the short-time propagator is taken to be the matrix element of the time evolution operator for a small time interval ε:

$$K(x_j, x_{j-1}, \varepsilon) = \langle x_j[\exp(-(\mathrm{i}/\hbar)H(p, x)\varepsilon]|x_{j-1}\rangle \simeq \langle x_j|1 - (\mathrm{i}/\hbar)H(p, x)\varepsilon\}|x_{j-1}\rangle.$$

In the following discussion we will not restrict the short-time propagator to be of the Lagrangian or Hamiltonian type. Our treatment will be applicable to both formulations. All we need to assume is that the short-time propagator for the free motion on a homogeneous manifold $\mathcal{M}$ is invariant under translations: $K(x_j, x_{j-1}, \varepsilon) = K(x_j', x_{j-1}', \varepsilon)$, where $x_j' = gx_j$, $\forall j = 0, 1, \ldots, N$. Choosing a fixed point $\boldsymbol{a}$ on $\mathcal{M}$, an arbitrary $x_j \in \mathcal{M}$ may be obtained through a local transformation:

$$x_j = g_j \boldsymbol{a} \qquad g_j \in \mathrm{G}. \tag{2}$$

Local means that for each short-time variable x_j a different group transformation g_j has to be used. As already mentioned, the stability group H is the subgroup of G leaving the fixed vector $\boldsymbol{a}$ invariant: $h\boldsymbol{a} = \boldsymbol{a}$, $\forall h \in \mathrm{H} \subset \mathrm{G}$. The homogeneous manifold $\mathcal{M}$ can be identified with the coset space G/H. With this construction, the short-time propagator may be viewed as a function on the group manifold of G; $K(x_j, x_{j-1}, \varepsilon) = K(g_j, g_{j-1}, \varepsilon)$. From the translation invariance it follows that this is a function of the combination $g_{j-1}^{-1}g_j$ only:

$$K(g_j, g_{j-1}, \varepsilon) = K(gg_j, gg_{j-1}, \varepsilon) = K(g_{j-1}^{-1}g_j, \varepsilon). \tag{3}$$

On the other hand, multiplying any group element g_j in (2) from the right with an arbitrary element of the stationary group H gives rise to the same x_j. Obviously, the short-time propagator (3) is invariant with respect to left and right multiplication by an arbitrary element of the subgroup H:

$$K(g, \varepsilon) = K(h_1^{-1}gh_2, \varepsilon) \qquad h_1, h_2 \in \mathrm{H}. \tag{4}$$

This property is the defining equation of the so-called zonal spherical functions [8].

Let us consider a unitary irreducible representation l of the group G, which associates with each element $g \in \mathrm{G}$ a unitary operator $\mathcal{D}^l(g)$ in a Hilbert space $\mathcal{H}^l(\mathrm{G/H})$. Introducing a basis $\{\boldsymbol{b}_k\}$, $k = 0, 1, 2, \ldots, \dim(\mathcal{H}^l - 1)$, in this space the matrix elements of the representation l are given by $\mathcal{D}^l_{mn}(g) = \langle \boldsymbol{b}_m|\mathcal{D}^l(g)|\boldsymbol{b}_m\rangle$. Furthermore, it is known that $\mathcal{H}^l$ contains a vector $|\boldsymbol{a}\rangle$ that is invariant under H, i.e. $\mathcal{D}^l(h)|\boldsymbol{a}\rangle = |\boldsymbol{a}\rangle$, $\forall h \in \mathrm{H}$. The subspace of $\mathcal{H}^l$ defined by this condition is one dimensional [9]. Choosing the basis $\{\boldsymbol{b}_k\}$ in such a way that $|\boldsymbol{b}_0\rangle = |\boldsymbol{a}\rangle$, the matrix elements $\mathcal{D}^l_{00}(g) = \langle \boldsymbol{a}|\mathcal{D}^l(g)|\boldsymbol{a}\rangle$ are the zonal spherical functions of the representation l. The functions $\mathcal{D}^l_{00}(g)$ form a complete set for any zonal spherical function in the Hilbert space $\mathcal{H}(\mathrm{G/H}) = \oplus_l \mathcal{H}^l$. In other words, any function $f(g)$, constant on the two-sided coset (H\G/H), can be decomposed in zonal spherical functions of unitary irreducible representations [10]:

$$f(g) = \sum\!\!\!\!\!\!\int_l d_l f_l \mathcal{D}^l_{00}(g) \qquad f_l = \int_{\mathrm{G}} f(g) \mathcal{D}^{l*}_{00}(g)\, \mathrm{d}g. \tag{5}$$

In the above, $\sum\!\!\!\!\!\!\int_l$ stands for the orthogonal sum of non-equivalent unitary irreducible representations. The question regarding which representations have to be encountered in this sum is in principle answered by the Frobenius theorem [11]. The number d_l is defined by the orthogonality relation $\int_G \mathscr{D}^l_{00}(g)\mathscr{D}^{l'*}_{00}(g)\,\mathrm{d}g = \delta(l', l)/d_l$, where $\delta(l', l) = \delta_{l'l}$ for discrete and $\delta(l', l) = \delta(l'-l)$ for continuous representation ('angular momentum') labels, respectively. For compact groups d_l is the dimension of the corresponding representation. Unitary irreducible representations of non-compact groups are infinite dimensional. However, we may call d_l the 'dimension' in this case, too.

We have already mentioned that the short-time propagator (4) is a zonal spherical function. Consequently, it may be decomposed into

$$K(g_{j-1}^{-1}g_j, \varepsilon) = \sum\!\!\!\!\!\!\int_l d_l f_l(\varepsilon)\mathscr{D}^l_{00}(g_{j-1}^{-1}g_j). \tag{6}$$

The Fourier coefficient is given by

$$f_l(\varepsilon) = \int_G K(g, \varepsilon)\mathscr{D}^{l*}_{00}(g)\,\mathrm{d}g. \tag{7}$$

Furthermore, through relation (2) we may identify the coordinates of $\boldsymbol{x}_j$ with those parameters of the group element g_j which do not belong to the subgroup H. Therefore, the volume element $\mathrm{d}\boldsymbol{x}_j$ appearing in the path integral (1), may be changed into the normalised Haar measure of the group G by multiplication with the identity $1 = \int_H \mathrm{d}h$ [9]:

$$\mathrm{d}\boldsymbol{x}_j = |\mathscr{M}| \int_H \mathrm{d}g_j. \tag{8}$$

The volume $|\mathscr{M}|$ of the space appears because of the relation $|\mathscr{M}| = \int_{\mathscr{M}} \mathrm{d}\boldsymbol{x} = |\mathscr{M}|\int_G \mathrm{d}g$. Formally, it may be viewed as the 'Jacobian' $\partial(\boldsymbol{x})/\partial(g) = |\mathscr{M}|$.

Inserting everything into the path integral, we obtain

$$K(\boldsymbol{x}_b, \boldsymbol{x}_a, T) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \left(\sum\!\!\!\!\!\!\int_{l_j} d_{l_j} f_{l_j}(\varepsilon)\mathscr{D}^{l_j}_{00}(g_{j-1}^{-1}g_j)\right) \prod_{j=1}^{N-1} |\mathscr{M}|\,\mathrm{d}g_j. \tag{9}$$

Making use of the orthogonality relation

$$\int_G \mathscr{D}^{l_j}_{00}(g_{j-1}^{-1}g_j)\mathscr{D}^{l_{j+1}}_{00}(g_j^{-1}g_{j+1})\,\mathrm{d}g_j = \frac{\delta(l_j, l_{j+1})}{d_{l_j}}\mathscr{D}^{l_j}_{00}(g_{j-1}^{-1}g_{j+1}) \tag{10}$$

the integration in (9) is easily performed and yields

$$K(\boldsymbol{x}_b, \boldsymbol{x}_a, T) = \sum\!\!\!\!\!\!\int_l \left(\lim_{N\to\infty} [f_l(\varepsilon)|\mathscr{M}|]^N\right)\frac{d_l}{|\mathscr{M}|}\mathscr{D}^l_{00}(g_0^{-1}g_N). \tag{11}$$

With the group composition law $\mathscr{D}^l_{00}(g_0^{-1}g_N) = \Sigma_m \mathscr{D}^l_{m0}(g_N)\mathscr{D}^{l*}_{m0}(g_0)$, the propagator may be written in the standard form

$$K(\boldsymbol{x}_b, \boldsymbol{x}_a, T) = \sum\!\!\!\!\!\!\int_l \exp[-(\mathrm{i}/\hbar)E_l T] \sum_m \Psi_{lm}(\boldsymbol{x}_b)\Psi^*_{lm}(\boldsymbol{x}_a) \tag{12}$$

where

$$E_l = (\mathrm{i}\hbar/T)\ln\left(\lim_{N\to\infty} [f_l(T/N)|\mathscr{M}|]^N\right) \tag{13}$$

$$\Psi_{lm}(\boldsymbol{x}) = \sqrt{d_l/|\mathscr{M}|}\,\mathscr{D}^l_{m0}(g) \tag{14}$$

are the energy spectrum and normalised wavefunctions, respectively. The matrix elements $\mathscr{D}^l_{m0}(g)$ are called associate spherical functions [8, 10]. They are eigenfunctions of the Laplace–Beltrami operator on $\mathscr{M} = \mathrm{G/H}$. The limit in (13) can be calculated from the Taylor expansion

$$\lim_{N\to\infty}[|\mathscr{M}|f_l(T/N)]^N = \lim_{N\to\infty}\left[|\mathscr{M}|f_l(0)\left(1+\frac{\dot f_l(0)T}{f_l(0)N}\right)\right]^N = \exp\left(\frac{\dot f_l(0)}{f_l(0)}T\right)$$

where $\dot f_l(\varepsilon) = (\mathrm{d}/\mathrm{d}\varepsilon)f_l(\varepsilon)$. In the last step we have made use of the normalisation $K(\boldsymbol{x}, \boldsymbol{a}, 0) = \delta(\boldsymbol{x}-\boldsymbol{a}) = \delta(g)/|\mathscr{M}|$ to find $f_l(0) = 1/|M|$. Hence, the energy spectrum is given by the time derivative of the Fourier coefficient at $\varepsilon = 0$:

$$E_l = \mathrm{i}\hbar|\mathscr{M}|\dot f_l(0). \tag{15}$$

It is worth mentioning that if the Fourier coefficients are given in the form $f_l(\varepsilon) = |\mathscr{M}|^{-1}\exp[-(\mathrm{i}/\hbar)E_l\varepsilon]$, the short-time propagator and the finite-time propagator are of the same form. Another point worth mentioning is the following. In (11) we have seen that the propagator is a zonal spherical function depending on the group element $g_0^{-1}g_N$. It is known that zonal functions on rank-one spaces depend only on one variable. This parameter may be identified with the geodesic distance s between the initial and final position. On the other hand, the classical action is given by $S_{\mathrm{cl}} = ms^2/2T$ and therefore the propagator depends on the classical action only.

As an example we consider the relativistic spinless free particle in n dimensions described by the Hamiltonian $H = c(\boldsymbol{p}^2 + m^2c^2)^{1/2}$. In the Hamiltonian form the short-time propagator has been calculated by Fukutaka and Kashiwa [3] and may be expressed in terms of modified Bessel functions of the third kind,

$$K(\boldsymbol{x}_j, \boldsymbol{x}_{j-1}, \varepsilon) = 2\mathrm{i}c\varepsilon\left(\frac{mc}{2\pi\mathrm{i}\hbar}\right)^{(n+1)/2}\frac{K_{(n+1)/2}\{(\mathrm{i}mc/\hbar)[c^2\varepsilon^2-(\Delta\boldsymbol{x}_j)^2]^{1/2}\}}{[c^2\varepsilon^2-(\Delta\boldsymbol{x}_j)^2]^{(n+1)/4}}. \tag{16}$$

The transformation group of R^n is the Euclidean group in n dimensions, being the semi-direct product of translations and rotations, $\mathrm{G} = \mathrm{T}^n \circledS \mathrm{SO}(n)$. Taking the fixed vector $\boldsymbol{a}$ to be the origin, the stationary group is $\mathrm{H} = \mathrm{SO}(n)$. The corresponding zonal spherical functions are given by Bessel functions [10], $\mathscr{D}^k_{00}(g) = \Gamma(n/2)(2/kr)^{(n-2)/2}J_{(n-2)/2}(kr)$. Here r is the radial polar coordinate of the translation vector in the group element g. For $g = g_{j-1}^{-1}g_j$ we have $r = |\Delta\boldsymbol{x}_j|$. The representation label k is related to the conserved linear momentum, $|\boldsymbol{p}| = \hbar k$. With the 'dimension' [12] $d_k = k^{n-1}/[2^{n-1}\pi^{n/2}\Gamma(n/2)]$ the Fourier coefficient is found to be $f_k(\varepsilon) = \exp[-(\mathrm{i}\varepsilon c/\hbar)(m^2c^2+\hbar^2k^2)^{1/2}]$ and the correct energy spectrum is obtained, $E_k = c(m^2c^2+\hbar^2k^2)^{1/2}$. Note that the invariant measure on R^n is identical with the Haar measure of T^n, i.e. $|R^n| = 1$. Due to the exponential form of $f_l(\varepsilon)$, the finite-time propagator has the same form as the short-time propagator (16) and may be expressed as a hypergeometric series

$$K(\boldsymbol{x}_b, \boldsymbol{x}_a, T) = \gamma\left(\frac{m\gamma}{2\pi\mathrm{i}\hbar T}\right)^{n/2}\exp\left(\frac{\mathrm{i}}{\hbar}S_{\mathrm{cl}}\right){}_2F_0(-n/2, (n+2)/2; \mathrm{i}\hbar\gamma/2mc^2T) \tag{17}$$

where $S_{\mathrm{cl}} = -mc^2T/\gamma$ is the classical action and $\gamma = [1-|\boldsymbol{x}_b-\boldsymbol{x}_a|^2/c^2T^2]^{-1/2}$. It is an interesting fact that the semiclassical approximation becomes exact for 'space dimension' $n = -2$ and 0.

Other examples, which can be put into this general formalism, are the non-relativistic quantum propagators in n-dimensional spaces of constant positive and negative curvature, which may be identified with the coset spaces $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ and

SO(n, 1)/SO(n), respectively [4]. The ordinary free particle in flat space can also be treated in this way [12]. A detailed discussion will be published elsewhere [13].

The present formalism is also applicable to the path integration on group spaces itself. In this case the subgroup reduces to the identity, $\mathrm{H}=\{e\}$. The corresponding zonal spherical functions are the group characters of unitary irreducible representations, $\chi^l(g)=\mathrm{Tr}\,\mathscr{D}^l(g)$ [8]. The result is the same as that obtained by Dowker [1].

In this letter we have presented a general prescription for the explicit path integration of a system evolving freely in a homogeneous space. However, the present procedure is also applicable to the general problem including an external potential. Here the translation invariance is broken by the potential. However, the expansion in zonal spherical functions is still applicable to that part of the short-time propagator which contains only the kinetic energy. For potentials with a remaining space symmetry the path integration can be performed partially. For example, in a spherical symmetric problem the angular integration can immediately be done using the above method. The remaining part of the short-time propagator is still a function on the group manifold and the general Fourier decomposition can be applied. However, the zonal spherical functions are no longer a complete set and the contributions of the associate spherical functions (14) have to be included. In this case the path integral does not lead to a simple form like (12) but gives rise to a perturbative expansion for the propagator.

It would also be interesting to consider the problem where the transformation group is broken into a discrete subgroup. This happens in the study of quantum chaos where the motion on a compact space of constant negative curvature is investigated [5].

For systems having a dynamical symmetry the present formalism can be used. By introducing additional space dimensions the path integral can be changed into that of a free particle moving on the dynamical group manifold [4]. Then one may proceed as described above for the special case with $\mathrm{H}=\{e\}$.

In a final remark we would like to mention that the expansion in unitary irreducible group representations is not only useful for the path integral evaluation of ordinary quantum mechanics, but may also be applied in the path integral formulation of field theories. Indeed, in pure lattice gauge theories the expansion in group characters is used extensively. However, our method can also be applied to gauge–Higgs models and lattice fermions [14].

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